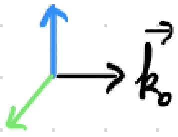
11) Mie scattering

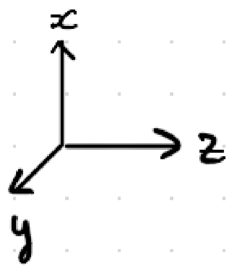
1. Problem statement

Scattering from a dielectric sphere (Mie, 1908)

$\vec{E}_{inc} = E_0 \vec{e}_x e^{-jk_0 z}$



$\vec{H}_{inc} = \frac{E_0}{\eta_0} \vec{e}_y e^{-jk_0 z}$



* Find the scattering cross sections σ_{scat}, σ_{abs}, σ_{ext}
* Find the fields everywhere, $\vec{E}_{scat}$, $\vec{E}_1$

$$\vec{E} = \begin{cases} \vec{E}_1 & r < a \\ \vec{E}_{inc} + \vec{E}_{scat} & r > a \end{cases}$$

* Boundary conditions :

$$\begin{cases} (E_i^\theta + E_s^\theta)\big|_{r=a} = E_1^\theta\big|_{r=a} \\ (H_i^\theta + H_s^\theta)\big|_{r=a} = H_1^\theta\big|_{r=a} \end{cases} \qquad \begin{cases} (E_i^\varphi + E_s^\varphi)\big|_{r=a} = E_1^\varphi\big|_{r=a} \\ (H_i^\varphi + H_s^\varphi)\big|_{r=a} = H_1^\varphi\big|_{r=a} \end{cases}$$

$\Rightarrow$ spherical fied expansions

2. Spherical field expansion

In a homogeneous isotropic medium, $\vec{\mathcal{E}}, \vec{\mathcal{D}}, \vec{\mathcal{H}}, \vec{\mathcal{B}}$ or $\vec{\mathcal{A}}$ satisfy

$$\vec{\nabla}(\vec{\nabla}.\vec{\mathcal{E}}) - \vec{\nabla}\times\vec{\nabla}\times\vec{\mathcal{E}} + k^2\vec{\mathcal{E}} = \vec{0} \quad (k^2 = \omega^2\mu\varepsilon).$$ vector wave equation

Theorem: If ψ is a solution of the scalar wave equation $\nabla^2\psi + k^2\psi = 0$ then one can construct 3 independent solutions of the vector wave equation:

$$\vec{L} = \vec{\nabla}\psi \qquad \vec{M} = \vec{\nabla}\times\vec{r}\psi = \vec{L}\times\vec{r} \qquad \vec{N} = \frac{1}{k}\vec{\nabla}\times\vec{M}$$

with $\vec{r} = r\vec{e_r} = \sqrt{x^2+y^2+z^2}\,\vec{e_r}$. [proof in Appendix I & III]

In spherical coordinates, we get a family of solutions from the scalar solutions

$$\psi_{\substack{e\\o}mn} = \cos m\varphi \; P_n^m(\cos\theta)\, z_n(kr)$$ [see App. II]

which we denote $\vec{M}_{\substack{e\\o}mn}$, $\vec{N}_{\substack{e\\o}mn}$ and $\vec{L}_{\substack{e\\o}mn}$. [see complete expressions in Ap. III]

Here P_n^m are associated Legendre polynomials and z_n is a spherical Bessel function j_n, y_n or a spherical Hankel function $h_n^{(1)}, h_n^{(2)}$.

Any solution of Maxwell's equation admits an expansion in terms of these vector fields, in particular the incident plane waves

$$\boxed{\vec{E}_i = E_0\,\vec{e_x}\,e^{-jk_0 z} = \sum_{m=1}^{+\infty} E_0\, j^{-m}\frac{2m+1}{m(m+1)}\left(\vec{M}^{(1)}_{o1m} + j\vec{N}^{(1)}_{e1m}\right)}$$

odd ≡ TE   even : TM

$$\boxed{\vec{H}_i = \frac{E_0}{Z}\,\vec{e_y}\,e^{-jk_0 z} = \sum_{m=1}^{+\infty} -\frac{E_0}{Z}\, j^{-m}\frac{2m+1}{m(m+1)}\left(\vec{M}^{(1)}_{e1m} - j\vec{N}^{(1)}_{e1m}\right)}$$

where $^{(1)}$ means $z_m = j_m$ and $^{(3)}$ will be used for $z_m = h_m^{(1)}$

[[We remind here the expressions of $\vec{M}_{\substack{e\\o}1n}$ & $\vec{N}_{\substack{e\\o}1n}$ derived in Appendix III

$$\boxed{\begin{aligned}
\vec{M}_{\substack{e\\o}1n} &= \left[\begin{smallmatrix}-\sin\\ +\cos\end{smallmatrix}\varphi \ \frac{P_n^1(\cos\theta)}{\sin\theta}\ z_n(\rho)\right]\vec{e_\theta} - \left[\begin{smallmatrix}\cos\\ \sin\end{smallmatrix}\varphi \ \frac{dP_n^1}{d\theta}(\cos\theta)\, z_n(\rho)\right]\vec{e_\varphi} \\
\vec{N}_{\substack{e\\o}1n} &= \left[\frac{n(n+1)}{\rho}\, z_n(\rho)\, P_n^1(\cos\theta)\ \begin{smallmatrix}\cos\\ \sin\end{smallmatrix}\varphi\right]\vec{e_r} \\
&\quad + \left[\frac{1}{\rho}\,\hat{z}_n'(\rho)\,\frac{\partial}{\partial\theta} P_n^1(\cos\theta)\ \begin{smallmatrix}\cos\\ \sin\end{smallmatrix}\varphi\right]\vec{e_\theta} \\
&\quad \mp \left[\frac{1}{\rho\sin\theta}\,\hat{z}_n'(\rho)\, P_n^1(\cos\theta)\ \begin{smallmatrix}\sin\\ \cos\end{smallmatrix}\varphi\right]\vec{e_\varphi}
\end{aligned}}$$

with $\rho = k_0 r$, $\hat{z}_n'(\rho) = \frac{\partial}{\partial\rho}\left[\rho\, z_n(\rho)\right]$]]

We thus look for the following fields: $\left(\alpha_n = j^{-n} \frac{2n+1}{n(n+1)}\right)$

$$\boxed{\begin{cases} \vec{E}_i = \sum_{n=1}^{+\infty} E_0 \alpha_n \left(\vec{M}^{(1)}_{o1n} + j \vec{N}^{(1)}_{e1n} \right) \\ \vec{E}_s = \sum_{n=1}^{+\infty} E_0 \alpha_n \left(c_n^{TE} \vec{M}^{(3)}_{o1n} + j c_n^{TM} \vec{N}^{(3)}_{e1n} \right) \\ \vec{E}_1 = \sum_{n=1}^{+\infty} E_0 \alpha_n \left(d_n^{TE} \vec{M}^{(1)}_{o1n} + j d_n^{TM} N^{(1)}_{e1n} \right) \\ \vec{H}_i = \sum_{n=1}^{+\infty} -\frac{E_0}{\eta_0} \alpha_n \left(\vec{M}^{(1)}_{e1n} - j \vec{N}^{(1)}_{o1n} \right) \\ \vec{H}_s = \sum_{n=1}^{+\infty} -\frac{E_0}{\eta_0} \alpha_n \left(c_n^{TM} \vec{M}^{(3)}_{e1n} - j c_n^{TE} \vec{N}^{(3)}_{o1n} \right) \\ \vec{H}_1 = \sum_{n=1}^{+\infty} -\frac{E_0}{\eta} \alpha_n \left(d_n^{TM} \vec{M}^{(1)}_{e1n} - j d_n^{TE} \vec{N}_{o1n} \right) \end{cases}}$$

Rk: ① Since the boundary condition must be satisfied, $\vec{E}_s$ and $\vec{E}_1$ can only have the same spherical field basis functions as $\vec{E}_i$. Same for $\vec{H}$.

② We keep the j factors in front of $\vec{N}$ so that it disappears when writing the BC

3. Boundary conditions

We enforce the boundary conditions noting that:

- we get four equations per harmonic n for the BC along θ, and 4 along φ
- these 4 equations are obtained by equating terms with the same angular dependence, which is the same as writing them for TE/TM alone
- $\Rightarrow$ only remain the radial part.
- The radial part for the θ and φ components of a given TE/TM component is the same $\Rightarrow$ only 4 equations are independent (we can only enforce the BC's related to θ).
- Looking at the radial parts of the TM waves

$$\begin{cases} \text{continuity of } E^{\theta} \Rightarrow \text{ of } \vec{N}_{e1m} \cdot \vec{e}_\theta \Rightarrow \dfrac{\hat{z}_m'^{TM}(\rho)}{\rho} \text{ continuous} \Rightarrow \\ \text{continuity of } H^{\theta} \Rightarrow \text{ of } \vec{M}_{e1m} \cdot \vec{e}_\theta \Rightarrow \dfrac{z_m^{TM}(\rho)}{\eta} \text{ continuous} \Rightarrow \end{cases} \boxed{\begin{matrix} \dfrac{\hat{z}_m'^{TM}(kr)}{k} \text{ continuous} \\ \dfrac{z_m^{TM}(kr)}{\eta} \text{ continuous} \end{matrix}}$$

- Looking at the radial parts of the TE waves

$$\begin{cases} \text{continuity of } E^{\theta} \Rightarrow \text{ of } \vec{M}_{o1m} \cdot \vec{e}_\theta \Rightarrow z_m^{TE} \text{ continuous} \Rightarrow \\ \text{continuity of } H^{\theta} \Rightarrow \text{ of } \vec{N}_{o1m} \cdot \vec{e}_\theta \Rightarrow \dfrac{\hat{z}_m'^{TE}}{\rho\,\eta} \text{ continuous} \end{cases} \boxed{\begin{matrix} z_m^{TE}(kr) \text{ continuous} \\ \dfrac{\hat{z}_m'^{TE}(kr)}{\mu} \text{ continuous} \end{matrix}}$$

with $\boxed{z_m^{TM}(\rho) = \begin{cases} j_m(k_0 r) + c_m^{TM} h_m^{(1)}(k_0 r) & r > a \\ d_m^{TM} j_m(kr) & r \leq a \end{cases}}$ $\boxed{z_m^{TE}(\rho) = \begin{cases} j_m(k_0 r) + c_m^{TE} h_m^{(1)}(k_0 r), & r > a \\ d_m^{TE} j_m(kr) & r \leq a \end{cases}}$

4. Scattering coefficients

TM case: $\begin{cases} \frac{1}{k} d_m^{TM} \hat{j}_m'(ka) = \frac{1}{k_0}\hat{j}_m'(k_0 a) + \frac{1}{k_0} c_m^{TM} \hat{h}_m^{(1)\prime}(k_0 a) \\ d_m^{TM} \frac{1}{\eta} j_m(ka) = \frac{1}{\eta_0} j_m(k_0 a) + \frac{1}{\eta_0} c_m^{TM} h_m^{(1)}(k_0 a) \end{cases}$

$$\Rightarrow \begin{pmatrix} \frac{\hat{j}_m'(ka)}{k} & -\frac{\hat{h}_m^{(1)\prime}(k_0 a)}{k_0} \\ \frac{j_m(ka)}{\eta} & -\frac{h_m^{(1)}(k_0 a)}{\eta_0} \end{pmatrix} \begin{pmatrix} d_m^{TM} \\ c_m^{TM} \end{pmatrix} = \begin{pmatrix} \frac{\hat{j}_m'(k_0 a)}{k_0} \\ \frac{j_m(k_0 a)}{\eta_0} \end{pmatrix}$$

Cramers rule:

$$c_m^{TM} = \frac{\begin{vmatrix} \frac{\hat{j}_m'(ka)}{k/\eta} & \frac{\hat{j}_m'(k_0 a)}{k_0/\eta_0} \\ \frac{j_m(ka)}{\cancel{\eta}\,1} & \frac{j_m(k_0 a)}{\cancel{\eta_0}\,1} \end{vmatrix}}{\begin{vmatrix} \frac{\hat{j}_m'(ka)}{k/\eta} & -\frac{\hat{h}_m^{(1)\prime}(k_0 a)}{k_0/\eta_0} \\ \frac{j_m(ka)}{\cancel{\eta}\,1} & -\frac{h_m^{(1)}(k_0 a)}{\cancel{\eta_0}\,1} \end{vmatrix}} \underset{\frac{k}{\eta}=\omega\varepsilon}{=} - \frac{\overbrace{\begin{vmatrix} j_m(ka) & j_m(k_0 a) \\ \frac{\hat{j}_m'(ka)}{\varepsilon} & \frac{\hat{j}_m'(k_0 a)}{\varepsilon_0} \end{vmatrix}}^{U_m^{TM}}}{\begin{vmatrix} j_m(ka) & j_m(k_0 a) - j\, n_m(k_0 a) \\ \frac{\hat{j}_m'(ka)}{\varepsilon} & \frac{\hat{j}_m'(k_0 a)}{\varepsilon_0} - j\,\frac{\hat{n}_m'(k_0 a)}{\varepsilon_0} \end{vmatrix}} \Rightarrow \boxed{c_m^{TM} = -\frac{U_m^{TM}}{U_m^{TM} - j V_m^{TM}}}$$

swap

divide by η divide by η_0

V_m^{TM} is like U_m^{TM} but with n on 2^{nd} column

Remarks: 1) lossless $\Rightarrow U_m^{TM}, V_m^{TM} \in \mathbb{R} \Rightarrow 0 \leq |c_m^{TM}| \leq 1$ *

2) $U_m = 0 \Leftrightarrow$ scattering zero (no scattering)
3) $V_m = 0 \Leftrightarrow C_m = -1 \Leftrightarrow$ scattering resonance
4) $U_m - jV_m = 0 \Leftrightarrow$ scattering pole = natural modes with $f = f_R + j f_I$ ($f_I \geq 0$ if passive)
5) $U_m = V_m = 0 \Leftrightarrow$ bound state in continuum

* this condition is too loose, in reality, c_m^{TM} belong to the disk of radius $\frac{1}{2}$ and center $-\frac{1}{2}$ in the complex plane, see PRB 89, 045122 (2014)

5) d_m^{TM} = exercise

TE case

$$\begin{cases} d_m^{TE}\, j_m(ka) = j_m(k_0 a) + c_m^{TE}\, h_m^{(1)}(k_0 a) \\ d_m^{TE}\, \dfrac{\hat{j}_m'(ka)}{\mu} = \dfrac{\hat{j}_m'(k_0 a)}{\mu_0} + c_m^{TE}\, \dfrac{\hat{h}_m^{(1)\prime}(k_0 a)}{\mu_0} \end{cases}$$

$$\Rightarrow \begin{pmatrix} j_m(ka) & -h_m^{(1)}(k_0 a) \\ \dfrac{\hat{j}_m'(ka)}{\mu} & -\dfrac{\hat{h}_m^{(1)\prime}(k_0 a)}{\mu} \end{pmatrix} \begin{pmatrix} d_m^{TE} \\ c_m^{TE} \end{pmatrix} = \begin{pmatrix} j_m(k_0 a) \\ \dfrac{\hat{j}_m'(k_0 a)}{\mu_0} \end{pmatrix}$$

$$\Rightarrow \boxed{c_m^{TE} = -\frac{U_m^{TE}}{U_m^{TE} - j V_m^{TE}}} \quad \text{with} \quad \left| \begin{matrix} U_m^{TE} \\ V_m^{TE} \end{matrix} \right. = \left| \begin{matrix} U_m^{TM} \\ V_m^{TM} \end{matrix} \right. (\varepsilon \leftrightarrow \mu)$$

consistent with duality

5. Cross sections

$\boxed{\text{* We first establish very general formulas that are not restricted to scattering from spheres}}$

$$h_n^{(1)}(\rho) \underset{\rho\to+\infty}{\sim} j^{n+1}\frac{e^{-j\rho}}{\rho} \qquad \hat{h}_n^{(1)\prime}(\rho) = \frac{\partial}{\partial\rho}\left[\rho h_n^{(1)}(\rho)\right] \underset{\rho\to+\infty}{\sim} -j^{n+2} e^{-j\rho}$$

$\Rightarrow$ In the far field the $\vec{e_\theta}$ and $\vec{e_\varphi}$ components are in $\frac{1}{r}$, and dominate the $\vec{e_r}$ component $\sim \frac{1}{r^2}$: TEM waves.

$$\Rightarrow \boxed{\begin{aligned} \vec{E_s} &= E_0 \frac{e^{-jk_0 r}}{jk_0 r}\vec{X}(\theta,\varphi) \\ \vec{H_s} &= \frac{1}{\eta}(\vec{e_r}\times\vec{E_s}) = \frac{E_0}{\eta_0}\frac{e^{-jk_0 r}}{jk_0 r}\vec{e_r}\times\vec{X}(\theta,\varphi) \end{aligned}}$$

$\vec{X}(\theta,\varphi)$: angular dependence of scattering upon plane wave excitation along $\vec{z}$: $\vec{X} = X_\theta\vec{e_\theta} + X_\varphi\vec{e_\varphi}$

Scattering cross section

$$W_{scat} = \oint_{S_\infty} \frac{1}{2}\,\mathrm{Re}\left[\vec{E_s}\times\vec{H_s}^*\right]\cdot\vec{e_r}\, r^2\sin\theta\, d\theta\, d\varphi \quad \text{with}$$

$$\frac{1}{2}\vec{E_s}\times\vec{H_s}^* = \frac{|E_0|^2}{2\eta_0}\frac{1}{k_0^2 r^2}\vec{X}\times(\vec{e_r}\times\vec{X}^*) = \frac{|E_0|^2}{2\eta_0 k_0^2 r^2}\begin{pmatrix}0\\X_\theta\\X_\varphi\end{pmatrix}\times\begin{pmatrix}0\\-X_\varphi^*\\X_\theta^*\end{pmatrix} = \frac{|E_0|^2}{2\eta_0 k_0^2 r^2}|\vec{X}|^2\vec{e_r}$$

$$\Rightarrow \sigma_{scat} = \frac{W_{scat}}{S_{in}} = \frac{W_{scat}}{\frac{1}{2\eta_0}|E_0|^2} \Rightarrow \boxed{\sigma_s = \oint \underbrace{\frac{|\vec{X}(\theta,\varphi)|^2}{k_0^2}}_{\text{differential scattering cross-section}} \sin\theta\, d\theta\, d\varphi}$$

Extinction cross section . (Jones, 1955)

$$W_{ext} = -\frac{1}{2} \mathrm{Re} \oint d\vec{S} \cdot \left(\vec{E}_i \times \vec{H}_s^* + \vec{E}_s \times \vec{H}_i^* \right)$$

$$= -\frac{1}{2} \mathrm{Re} \iint \frac{|E_0|^2}{\eta_0} e^{-jk_0 z} \frac{e^{jk_0 r}}{-jk_0 r} \left(\vec{z} \times \vec{e}_r \times \vec{X}^* \right) \cdot \vec{e}_r \, r^2 \sin\theta \, d\theta \, d\varphi$$

$$- \frac{1}{2} \mathrm{Re} \iint \frac{|E_0|^2}{\eta_0} \frac{e^{-jk_0 r}}{jk_0 r} e^{jk_0 z} (\vec{X} \times \vec{y}) \cdot \vec{e}_r \, r^2 \sin\theta \, d\theta \, d\varphi$$

$$\Rightarrow \sigma_{ext} = - \mathrm{Re} \left\{ \frac{e^{jk_0 r}}{-jk_0 r} \iint e^{-jk_0 r \cos\theta} r^2 \sin\theta \, d\theta \, d\varphi \left[\vec{z} \times \vec{e}_r \times \vec{X}^* \right] \cdot \vec{e}_r \right.$$

$$\left. + \frac{e^{-jk_0 r}}{jk_0 r} \iint e^{jk_0 r \cos\theta} r^2 \sin\theta \, d\theta \, d\varphi \left[\vec{X} \times \vec{y} \right] \cdot \vec{e}_r \right.$$

Algebraic manipulations

① $\vec{A} \times (\vec{B} \times \vec{C}) = (\vec{A} \cdot \vec{C}) \vec{B} - (\vec{A} \cdot \vec{B}) \vec{C}$

$$\Rightarrow \vec{e}_r \cdot [\underbrace{\vec{x}}_{\vec{A}} \times \underbrace{\vec{e}_r}_{\vec{B}} \times \underbrace{\vec{X}^*}_{\vec{C}}] = \vec{e}_r \cdot \left[(\vec{x} \cdot \vec{X}^*) \vec{e}_r - (\vec{x} \cdot \vec{e}_r) \vec{X}^* \right] = \underline{\vec{x} \cdot \vec{X}^*}$$

② $[\vec{A} \times \vec{B}] \cdot [\vec{C} \times \vec{D}] = (\vec{A} \cdot \vec{C})(\vec{B} \cdot \vec{D}) - (\vec{A} \cdot \vec{D})(\vec{B} \cdot \vec{C})$

$$\Rightarrow \vec{e}_r \cdot [\vec{X} \times \vec{y}] = \vec{y} \cdot [\vec{e}_r \times \vec{X}] = [\underbrace{\vec{z}}_{\vec{A}} \times \underbrace{\vec{x}}_{\vec{B}}] \cdot [\underbrace{\vec{e}_r}_{\vec{C}} \times \underbrace{\vec{X}}_{\vec{D}}]$$

$$= (\vec{z} \cdot \vec{e}_r)(\vec{x} \cdot \vec{X}) - (\vec{z} \cdot \vec{X})(\vec{x} \cdot \vec{e}_r)$$

$$\Rightarrow \vec{e}_r \cdot [\vec{X} \times \vec{y}] = \underline{\cos\theta (\vec{x} \cdot \vec{X}) - \sin\theta \cos\varphi (\vec{z} \cdot \vec{X})}$$

$$\Rightarrow \sigma_{ext} = - \mathrm{Re} \left\{ \frac{e^{jk_0 r}}{-jk_0 r} \overbrace{\iint e^{-jk_0 r \cos\theta} (\vec{x} \cdot \vec{X}^*) r^2 \sin\theta \, d\theta \, d\varphi}^{I_1} \right.$$

$$+ \frac{e^{-jk_0 r}}{jk_0 r} \overbrace{\iint e^{jk_0 r \cos\theta} \cos\theta (\vec{x} \cdot \vec{X}) r^2 \sin\theta \, d\theta \, d\varphi}^{I_2}$$

$$\left. - \frac{e^{-jk_0 r}}{jk_0 r} \overbrace{\iint e^{jk_0 r \cos\theta} \sin\theta \cos\varphi (\vec{z} \cdot \vec{X}) r^2 \sin\theta \, d\theta \, d\varphi}^{I_3} \right\}$$

[this is eq. 3.23 in Bohren-Huffman's book on scattering]

We now perform the variable change $\mu = \cos\theta$ $\quad d\mu = -\sin\theta \, d\theta$

Integrals in I_1, I_2, I_3 are of the form $\quad r^2 \int_0^{2\pi} d\varphi \int_{-1}^{1} e^{-jk_0 r\mu} f(\mu) \, d\mu$.

An integration by part give $r^2 \left[\dfrac{e^{-jk_0 r} f(1) - e^{+jk_0 r} f(-1)}{-jk_0 r} + \underbrace{\int_{-1}^{1} \dfrac{e^{-jk_0 r\mu}}{-jk_0 r} \dfrac{df}{d\mu}}_{O\left(\frac{1}{k_0^2 r^2}\right)} \right] 2\pi$

if $\frac{df}{d\mu}$ is bounded

Thus, in the limit $k_0 r \to \infty$, only the first term matters (particular case of the method of stationary phase):

sign

$$I_1 = 2\pi r^2 \frac{e^{-jk_0 r} [\vec{x} . \vec{X}^*]_{\theta=0} - e^{jk_0 r} [\vec{x} . \vec{X}^*]_{\theta=\pi}}{-jk_0 r} \quad ; \quad I_2 = 2\pi r^2 \frac{e^{+jk_0 r} [\vec{x} . \vec{X}]_{\theta=0} + e^{-jk_0 r} [\vec{x} . \vec{X}]_{\theta=\pi}}{+jk_0 r}$$

$$I_3 = 0$$

$$\Rightarrow \sigma_{ext} = -\mathrm{Re}\left\{ -\overbrace{\frac{2\pi}{k_0^2}[\vec{x}.\vec{X}^*]_{\theta=0}}^{z_2^*} + \overbrace{\frac{2\pi}{k_0^2} e^{2jk_0 r} [\vec{x}.\vec{X}^*]_{\theta=\pi}}^{z_1} - \overbrace{\frac{2\pi}{k_0^2}[\vec{x}.\vec{X}]_{\theta=0}}^{z_2} - \underbrace{\frac{2\pi}{k_0^2} e^{-2jk_0 r} [\vec{x}.\vec{X}]_{\theta=\pi}}_{z_1^*} \right\} = -\mathrm{Re}\left\{ -\underbrace{(z_2 + z_2^*)}_{\in \mathbb{R}} + \underbrace{z_1 - z_1^*}_{\in j\mathbb{R}} \right\}$$

$$\Rightarrow \sigma_{ext} = 2\,\mathrm{Re}(z_2) \Rightarrow \boxed{\sigma_{ext} = \frac{4\pi}{k_0^2} \mathrm{Re}\, [\vec{x} . \vec{X}]_{\theta=0}}$$

optical theorem

only the shadow matters

Absorption cross section $\boxed{\sigma_{abs} = \sigma_{ext} - \sigma_{scat}}$

* Now we particularize the formulas of $\sigma_s, \sigma_{ext}, \sigma_d$ for spheres:

Calculation of $\vec{X}$:

$$\vec{E}_s = \sum_{m=1}^{+\infty} E_0\,\alpha_m \left(c_m^{TE}\,\vec{M}^{(3)}_{o1m} + j\,c_m^{TM}\,\vec{N}^{(3)}_{e1m} \right) \quad \text{with } \alpha_m = j^{-m}\frac{2m+1}{m(m+1)}$$

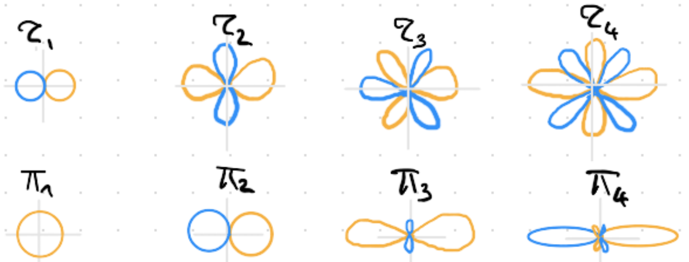
In far field, $h_m^{(1)}(\rho) \underset{\rho\to+\infty}{\sim} j^{m+2}\dfrac{e^{-j\rho}}{j\rho}$ and $\dfrac{\hat{h}_m^{(1)\prime}(\rho)}{\rho} \sim -j^{m+3}\dfrac{e^{-j\rho}}{j\rho}$

$$\Rightarrow \vec{E}_s = E_0\frac{e^{-jk_0 r}}{jk_0 r}\sum_{m=1}^{+\infty}\alpha_m\Big\{ c_m^{TE} j^{m+2}\cos\varphi\,\frac{P_m^1(\cos\theta)}{\sin\theta}\vec{e}_\theta - c_m^{TE} j^{m+2}\sin\varphi\,\frac{dP_m^1}{d\theta}(\cos\theta)\,\vec{e}_\varphi + c_m^{TM} j^{m+2}\cos\varphi\,\frac{dP_m^1}{d\theta}(\cos\theta)\,\vec{e}_\theta - c_m^{TM} j^{m+2}\sin\varphi\,\frac{P_m^1(\cos\theta)}{\sin\theta}\vec{e}_\varphi\Big\}$$

$$\Rightarrow \boxed{\vec{X}(\theta,\varphi) = -\sum_{m=1}^{+\infty}\frac{2m+1}{m(m+1)}\left\{ \vec{e}_\theta\cos\varphi\left[c_m^{TE}\frac{P_m^1(\cos\theta)}{\sin\theta} + c_m^{TM}\frac{dP_m^1}{d\theta}(\cos\theta)\right] + \vec{e}_\varphi\sin\varphi\left[-c_m^{TM}\frac{P_m^1(\cos\theta)}{\sin\theta} - c_m^{TE}\frac{dP_m^1}{d\theta}(\cos\theta)\right]\right\}}$$

Or, introducing the functions $\pi_m = \dfrac{P_m^1(\cos\theta)}{\sin\theta}$, $\tau_m = \dfrac{dP_m^1}{d\theta}(\cos\theta)$

$$\boxed{\vec{X}(\theta,\varphi) = \vec{e}_\theta\left\{-\cos\varphi\sum_{m=1}^{+\infty}\frac{2m+1}{m(m+1)}\left(c_m^{TE}\pi_m + c_m^{TM}\tau_m\right)\right\} + \vec{e}_\varphi\left\{\sin\varphi\sum_{m=1}^{+\infty}\frac{2m+1}{m(m+1)}\left(c_m^{TM}\pi_m + c_m^{TE}\tau_m\right)\right\}}$$



$\theta = \pi$, $\mu = -1$ $\longrightarrow$ $\theta = 0$, $\mu = 1$

m odd $\Rightarrow$ τ_m odd, π_m even with μ

m even $\Rightarrow$ opposite

$$P_1^1(\cos\theta) = \sin\theta \Rightarrow \tau_1 = +\cos\theta \text{ and } \pi_1 = 1$$

$$P_2^1(\cos\theta) = \frac{3}{2}\sin 2\theta \Rightarrow \tau_2 = +3\cos 2\theta \text{ and } \pi_2 = \frac{3}{2}\frac{\sin 2\theta}{\sin\theta} = \frac{3}{\cancel{2}}\frac{2\sin\theta\cos\theta}{\sin\theta} = 3\cos\theta$$

<u>ex</u>: c_1^{TM} alone $\Rightarrow \vec{E}_S \sim -\cos\theta\cos\varphi\,\vec{e_\theta} + \sin\varphi\,\vec{e_\varphi}$ $\Rightarrow$ electric dipole along $\vec{x}$

c_1^{TE} alone $\Rightarrow \vec{E}_S \sim -\cos\varphi\,\vec{e_\theta} + \cos\theta\sin\varphi\,\vec{e_\varphi}$ $\Rightarrow$ magnetic dipole along $\vec{y}$

[<u>exercise</u>: $\vec{J}(\vec{r}) = \vec{x}\,\delta(r)$ radiates $\vec{E}^{ff} = jk_0\eta_0 \frac{e^{-jkr}}{4\pi r}(\sin\varphi\,\vec{e_\varphi} - \cos\theta\cos\varphi\,\vec{e_\theta})$]

$\vec{M}(\vec{r}) = \vec{y}\,\delta(r)$ radiates $\vec{E}^{ff} = jk_0\eta_0\, g(r)\,(-\cos\varphi\,\vec{e_\theta} + \cos\theta\sin\varphi\,\vec{e_\varphi})$]

<u>polarisability</u>? for c_1^{TM} alone, $\vec{X}_S = \frac{3}{2}(-\cos\theta\cos\varphi\,\vec{e_\theta} + \sin\varphi\,\vec{e_\varphi})\,c_1^{TM}$

$$\Rightarrow \vec{E}_S = j E_0 \frac{e^{-jk_0 r}}{jk_0 r}\frac{3}{2}(+\cos\theta\cos\varphi\,\vec{e_\theta} \pm \sin\varphi\,\vec{e_\varphi})$$

When studying dipolar scattering, we found $\vec{E}_S = \omega p k_0 \eta_0 \frac{e^{-jk_0 r}}{4\pi r}(\cos\theta\cos\varphi\,\vec{e_\theta} - \sin\varphi\,e_\varphi)$

We deduce $\frac{\omega p k_0 \eta_0}{4\pi} = c_1^{TM} j\,\cancel{E_0} \times \frac{3}{2k_0}$ $\Rightarrow \alpha_e = j\frac{3}{\cancel{2}k_0} c_1^{TM} \frac{\overset{2}{\cancel{4}}\pi}{\omega k_0 \eta_0} = j\frac{6\pi\varepsilon_0}{k_0^3}c_1^{TM}$

$p = \alpha_e E_0$; $\omega \to k_0/\sqrt{\mu_0\varepsilon_0}$; $\eta_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}}$

$$\Rightarrow \boxed{\alpha_e = j\frac{6\pi\varepsilon_0}{k_0^3}c_1^{TM} \qquad \alpha_m = j\frac{6\pi\mu_0}{k_0^3}c_1^{TE}}$$

dipolar polarizabilities are proportional to scattering coefficients

<u>Differential scattering cross-section:</u> From $\vec{X}$ we calculate $\sigma_d = \frac{|X|^2}{k_0^2}$ as:

$$\boxed{\sigma_d k_0^2 = \cos^2\varphi \left| \sum_{n=1}^{+\infty} \frac{2n+1}{n(n+1)} \left(c_n^{TE} \pi_n + c_n^{TM} \tau_n \right) \right|^2 + \sin^2\varphi \left| \sum_{n=1}^{+\infty} \frac{2n+1}{n(n+1)} \left(c_n^{TM} \pi_n + c_n^{TE} \tau_n \right) \right|^2}$$

where $|z|^2 = z z^*$

<u>Scattering cross-section</u>

To get σ_s, we need to integrate over the sphere. Let's focus on the first term:

$$\underbrace{\int_0^{2\pi} \cos^2\varphi \, d\varphi}_{\frac{1}{2}\cdot 2\pi} \int_0^{\pi} \sum_{n=1}^{+\infty} \sum_{m=1}^{+\infty} \frac{2n+1}{n(n+1)} \frac{2m+1}{m(m+1)} \underbrace{\left(c_n^{TE} \pi_n + c_n^{TM} \tau_n \right) \left(c_m^{TE*} \pi_m + c_m^{TM*} \tau_m \right)}_{|c_n^{TE}|^2 \pi_n \pi_m + |c_n^{TM}|^2 \tau_n \tau_m + c_n^{TM} c_n^{TE*} \tau_n \pi_m + c_n^{TE} c_n^{TM*} \pi_n \tau_m} \sin\theta \, d\theta$$

The second term is similar:

$$\underbrace{\int_0^{2\pi} \sin^2\varphi \, d\varphi}_{1/2 \cdot 2\pi} \int_0^{\pi} \sum_{n=1}^{+\infty} \sum_{m=1}^{+\infty} \frac{2n+1}{n(n+1)} \frac{2m+1}{m(m+1)} \underbrace{\left(c_n^{TM} \pi_n + c_n^{TE} \tau_n \right) \left(c_m^{TM*} \pi_m + c_m^{TE*} \tau_m \right)}_{|c_n^{TM}|^2 \pi_n \pi_m + |c_n^{TE}|^2 \tau_n \tau_m + c_n^{TE} c_m^{TM*} \tau_n \pi_m + c_n^{TM} c_m^{TE*} \pi_n \tau_m} \sin\theta \, d\theta$$

$\Downarrow$

$$\sigma_s \frac{k_0^2}{2\pi} = \frac{1}{2} \sum_{n=1}^{+\infty} \sum_{m=1}^{+\infty} \frac{2n+1}{n(n+1)} \frac{2m+1}{m(m+1)} \int_0^{\pi} \sin\theta \, d\theta \Big\{ \left(|c_n^{TM}|^2 + |c_n^{TE}|^2 \right) \left(\pi_n \pi_m + \tau_n \tau_m \right)$$
$$+ \left(c_n^{TM} c_m^{TE*} + c_n^{TM*} c_m^{TE} \right) \left(\tau_n \pi_m + \pi_n \tau_m \right) \Big\}$$

From the orthogonality of Legendre functions we found

$$\int_0^{\pi} \sin\theta \, d\theta \left(\pi_n \pi_m + \tau_n \tau_m \right) = \delta_{nm} \frac{2n^2 (n+1)^2}{2n+1} \quad \text{and}$$

$$\int_0^{\pi} \sin\theta \, d\theta \left(\tau_n \pi_m + \pi_n \tau_m \right) = \int_0^{\pi} \left(\frac{dP_n^1}{d\theta} P_m^1 + P_n^1 \frac{dP_m^1}{d\theta} \right) d\theta$$
$$= \int_0^{\pi} \frac{d}{d\theta} \left(P_n^1 P_m^1 \right) d\theta = \left[P_n^1 P_m^1 \right]_0^{\pi} = 0$$

$$\Rightarrow \quad \boxed{\sigma_s = \frac{2\pi}{k_0^2} \sum_{n=1}^{+\infty} (2n+1) \left(|c_n^{TM}|^2 + |c_n^{TE}|^2 \right)}$$

Extinction cross section

$$\sigma_{ext} = \frac{4\pi}{k_0^2}\,\mathrm{Re}\left\{(\vec{X}\cdot\vec{x})_{\theta=0}\right\} = \frac{4\pi}{k_0^2}\,\mathrm{Re}\left\{\left(X_\theta(\vec{e}_\theta\cdot\vec{x}) + X_\varphi(\vec{e}_\varphi\cdot\vec{x})\right)_{\theta=0}\right\}, \text{ but since}$$

$$\vec{e}_\theta\cdot\vec{x} = \cos\theta\cos\varphi \;, \quad \vec{e}_\varphi\cdot\vec{x} = -\sin\varphi \;,$$

$$\vec{\sigma}_{ext} = \frac{4\pi}{k_0^2}\,\mathrm{Re}\left\{X_\theta(\theta=0)\cos\varphi - X_\varphi(\theta=0)\sin\varphi\right\} \quad \text{with}$$

$$\begin{cases} X_\theta(\theta=0) = -\cos\varphi \displaystyle\sum_{n=1}^{+\infty} \frac{2n+1}{n(n+1)}\left(c_n^{TE}\pi_n(\theta=0) + c_n^{TM}\tau_n(\theta=0)\right) \\ X_\varphi(\theta=0) = \sin\varphi \displaystyle\sum_{n=1}^{+\infty} \frac{2n+1}{n(n+1)}\left(c_n^{TM}\pi_n(\theta=0) + c_n^{TE}\tau_n(\theta=0)\right) \end{cases}$$

Since $\pi_n = \dfrac{P_n^1(\cos\theta)}{\sin\theta} = \dfrac{P_n^1(\mu)}{\sqrt{1-\mu^2}}$, $\tau_n = \dfrac{dP_n^1(\cos\theta)}{d\theta} = -\dfrac{dP_n^1(\mu)}{d\mu}\sqrt{1-\mu^2}$

still with $\mu = \cos\theta$, $d\mu = -\sin\theta\, d\theta$. From Rodrigues formula

$$P_n^1(\mu) = (1-\mu^2)^{1/2}\,\frac{dP_n(\mu)}{d\mu} \Rightarrow \pi_n(1) = \left.\frac{dP_n}{d\mu}\right|_{\mu=1}$$

But from the defining differential equation of Legendre polynomials

$$(1-\mu^2)\frac{d^2P_n}{d\mu^2} - 2\mu\frac{dP_n}{d\mu} + n(n+1)P_n = 0 \overset{\mu=1}{\Rightarrow} \left.\frac{dP_n}{d\mu}\right|_{\mu=1} = \frac{n(n+1)}{2}\cancelto{1}{P_n(1)}$$

For τ_n we use the identity : $\dfrac{dP_n^m}{d\theta} = -\sqrt{1-\mu^2}\,\dfrac{dP_n^m}{d\mu} = \dfrac{1}{2}\left[(n-m+1)(n+m)P_n^{m-1} - P_n^{m+1}\right]$

$$\Rightarrow \tau_n = \frac{n(n+1)}{2}P_n(\mu) - P_n^1(\mu) \Rightarrow \tau_n(1) = \frac{n(n+1)}{2}\cancelto{1}{P_n(1)} - \cancelto{0}{P_n^1(1)}$$

$$\Rightarrow \sigma_{ext} = \frac{4\pi}{k_0^2}\,\mathrm{Re}\left\{\left(-\frac{1}{2}\cos^2\varphi - \frac{1}{2}\sin^2\varphi\right)\sum_{n=1}^{+\infty}(2n+1)\left(c_n^{TE} + c_n^{TM}\right)\right\}$$

$$\Rightarrow \boxed{\sigma_{ext} = -\frac{2\pi}{k_0^2}\sum_{n=1}^{+\infty}(2n+1)\,\mathrm{Re}\left[c_n^{TM} + c_n^{TE}\right]}$$

<u>Backscattering</u> Since $\pi_m(-\mu) = (-1)^{m-1}\pi_m(\mu)$, $\tau_m(-\mu) = (-1)^m \tau_m(\mu)$

$$\pi_m(-1) = (-1)^{m-1}\pi_m(1) = (-1)^{m-1}\frac{m(m+1)}{2} = -(-1)^m \frac{m(m+1)}{2}$$

$$\tau_m(-1) = (-1)^m \frac{m(m+1)}{2} = -\pi_m(-1)$$

$$\Rightarrow \boxed{\sigma_d(\theta=\pi) = \frac{1}{4k_0^2}\left|\sum_{m=1}^{+\infty}(-1)^m(2m+1)\left(C_m^{TM} - C_m^{TE}\right)\right|^2}$$

is 0 if $C_m^{TM} = C_m^{TE}$ i.e $\varepsilon = \mu$ (matched object) is reduced when the dominant term m_0 satisfies $C_{m_0}^{TE} = C_{m_0}^{TM}$

6. Small particles

$$U_m^{TM} = \begin{vmatrix} j_m(ka) & j_m(k_0 a) \\ \frac{\hat{j}_m'(ka)}{\varepsilon_0 \varepsilon_r} & \frac{\hat{j}_m'(k_0 a)}{\varepsilon_0} \end{vmatrix} \qquad V_m^{TM} = \begin{vmatrix} j_m(ka) & y_m(k_0 a) \\ \frac{\hat{j}_m'(ka)}{\varepsilon_0 \varepsilon_r} & \frac{\hat{y}_m'(k_0 a)}{\varepsilon_0} \end{vmatrix}$$

Let's look at the quasistatic limit $x = ka \ll 1$, $x_0 = k_0 a \ll 1$ using the first term of the Taylor series of j_m and y_m (given in Appendix II, using $\Gamma(m+1) = m!$)

$$j_m(x) \underset{0}{\sim} 2^m x^m \frac{m!}{(2m+1)!} \qquad \hat{j}_m'(x) = \frac{d}{dx}[x j_m(x)] \underset{0}{\sim} 2^m x^m \frac{(m+1)!}{(2m+1)!}$$

$$y_m(x) \underset{0}{\sim} -\frac{1}{2^m x^{m+1}} \frac{(2m)!}{m!} \qquad \hat{y}_m'(x) = \frac{d}{dx}[x y_m(x)] \underset{0}{\sim} -\frac{1}{2^m}(-m)\frac{1}{x^{m+1}}\frac{(2m)!}{m!}$$
$$= \frac{1}{2^m x^{m+1}} \frac{(2m)!}{(m-1)!}$$

$$x = \sqrt{\varepsilon_r \mu_r}\, x_0$$

$$U_m^{TM} \underset{0}{\sim} \begin{vmatrix} 2^m x_0^m \sqrt{\varepsilon_r\mu_r}^m \frac{m!}{(2m+1)!} & 2^m x_0^m \frac{m!}{(2m+1)!} \\ 2^m x_0^m \frac{\sqrt{\varepsilon_r\mu_r}^m}{\varepsilon_0\varepsilon_r} \frac{(m+1)!}{(2m+1)!} & 2^m x_0^m \frac{1}{\varepsilon_0}\frac{(m+1)!}{(2m+1)!} \end{vmatrix} = 2^m x_0^m \frac{\sqrt{\varepsilon_r\mu_r}^m}{\varepsilon_0} \frac{m!}{(2m+1)!} 2^m x_0^m \frac{m!}{(2m+1)!} \times \underbrace{\begin{vmatrix} 1 & 1 \\ \frac{m+1}{\varepsilon_r} & m+1 \end{vmatrix}}_{(m+1)\left(1-\frac{1}{\varepsilon_r}\right)}$$

$$\Rightarrow \boxed{U_m^{TM} \underset{0}{\sim} x_0^{2m} \frac{\sqrt{\varepsilon_r\mu_r}^m}{\varepsilon_0} \left(\frac{2^m m!}{(2m+1)!}\right)^2 (m+1) \frac{\varepsilon_r - 1}{\varepsilon_r}}$$

$$V_m^{TM} \underset{0}{\sim} \begin{vmatrix} \cancel{2^m} \cancel{x_0^m} \sqrt{\varepsilon_r\mu_r}^m \frac{m!}{(2m+1)!} & -\frac{1}{\cancel{2^m} \cancel{x_0^{m+1}}_{x_0}} \frac{(2m)!}{m!} \\ \cancel{2^m} \cancel{x_0^m} \frac{\sqrt{\varepsilon_r\mu_r}^m}{\varepsilon_0\varepsilon_r} \frac{(m+1)!}{(2m+1)!} & \frac{1}{\cancel{2^m} \cancel{x_0^{m+1}}_{x_0} \varepsilon_0} \frac{(2m)!}{(m-1)!} \end{vmatrix} = \frac{\sqrt{\varepsilon_r\mu_r}^m}{\varepsilon_0 x_0} \frac{\cancel{m!}}{\cancel{(2m+1)!}} \frac{\cancel{(2m)!}}{\cancel{m!}} \times \begin{vmatrix} 1 & -1 \\ \frac{m+1}{\varepsilon_r} & m \end{vmatrix}$$

$$\Rightarrow \boxed{V_m^{TM} \underset{0}{\sim} \frac{\sqrt{\varepsilon_r\mu_r}^m}{x_0} \frac{1}{2m+1} \frac{m\varepsilon_r + m + 1}{\varepsilon_r}}$$

scattering resonances $\Leftrightarrow \begin{cases} V_m^{TM} = 0 \\ C_m^{TM} = -1 \end{cases} \Leftrightarrow \boxed{\varepsilon_r = -\frac{m+1}{m}}$

plasmonic resonances

$m=1$, $\varepsilon_r = -2 \rightarrow$ dipole

$m=2$, $\varepsilon_r = -\frac{3}{2} \rightarrow$ quadrupole

Far away from the plasmonic resonance $V_m^{TM} \sim \frac{1}{x_0}$ dominates the denominator :

$$C_m^{TM} \underset{0}{\sim} - \frac{U_m^{TM}}{-j V_m^{TM}} = -j\, x_0^{2m} \frac{\sqrt{\varepsilon_r \cancel{\mu_r}}^{\,m}}{\cancel{\varepsilon_0}} \left(\frac{2^m m!}{(2m+1)!} \right)^2 (m+1) \frac{\varepsilon_r - 1}{\cancel{\varepsilon_r}} \times x_0 \frac{1}{\sqrt{\cancel{\varepsilon_r \mu_r}}^{\,m}} (2m+1) \frac{\cancel{\varepsilon_r \varepsilon_0}}{m \varepsilon_r + m + 1}$$

$$\Rightarrow \boxed{C_m^{TM} \underset{0}{\sim} -j\, x_0^{2m+1} \frac{(2m+1)(m+1)\, 4^m (m!)^2}{(2m+1)!^2} \frac{\varepsilon_r - 1}{m \varepsilon_r + m + 1}}$$

Remarks :

1_ Goes to zero faster for higher m

2_ μ_r disappeared

3_ C_m^{TE} obtained by duality $\Rightarrow$ irrelevant for small non-magnetic spheres

4\. Small atmospheric particle look larger at high f $\Rightarrow$ blue is more scattered than red $\Rightarrow$ $\begin{cases} \text{red light} = \text{forward direction} \\ \text{blue light} = \text{scattered at all angles} \end{cases}$

explains red $\rightarrow$ blue colors at sunset.

$$\sigma_{scat} \simeq \frac{2\pi}{k_0^2} (2\times 1+1) \left[|C_1^{TM}|^2 + |C_1^{TE}|^2 \right]$$

$$\sigma_{scat} \simeq \frac{\cancel{6}^{2}\cancel{\pi}\lambda_0^2}{\cancel{k_0^2}_{\cancel{4}\pi^2}} x_0^6 \left(\frac{\cancel{6}\times \cancel{4}2}{(3\times \cancel{2})^2} \right)^2 \left[\left| \frac{\varepsilon_r - 1}{\varepsilon_r + 2} \right|^2 + \left| \frac{\mu_r - 1}{\mu_r + 2} \right|^2 \right]$$

$\hookrightarrow \cancel{4}/\cancel{9}3$

$$\boxed{\sigma_{scat} \simeq \frac{2\lambda_0^2}{3\pi} x_0^6 \left[\left| \underbrace{\frac{\varepsilon_r - 1}{\varepsilon_r + 2}}_{TM} \right|^2 + \left| \underbrace{\frac{\mu_r - 1}{\mu_r + 2}}_{TE} \right|^2 \right]}$$

for small particles away from plasmonic resonances

Kerker condition

$$\sigma_{ext} = -\frac{\lambda_0^2}{2\pi} \sum_{m=1}^{+\infty} (2m+1)\, \mathrm{Re}\left[C_m^{TE} + C_m^{TM} \right]$$

$$\sigma_s = \frac{\lambda_0^2}{2\pi} \sum_{m=1}^{+\infty} (2m+1) \left(|C_m^{TE}|^2 + |C_m^{TM}|^2 \right)$$

Claim by Kerker: For small particles $C_1^{TM} = -\frac{2j}{3} x_0^3 \frac{\varepsilon_r - 1}{\varepsilon_r + 2}$, $C_1^{TE} = -\frac{2j}{3} x_0^3 \frac{\mu_r - 1}{\mu_r + 2}$

if $C_1^{TM} = -C_1^{TE}$, I can have 0 forward scattering but non-zero scattering in other directions:

$$\frac{\varepsilon_r - 1}{\varepsilon_r + 2} = -\frac{\mu_r - 1}{\mu_r + 2} \Rightarrow \boxed{\varepsilon = \frac{4-\mu}{2\mu+1}}$$

Kerker condition

Actually, by the optical theorem, forward scattering cannot be zero:

$$C_1^{TM/TE} = \left(\frac{U_m^{TM} - jV_m^{TM}}{-U_m^{TM}} \right)^{-1} = \left(-1 + j\frac{V_m^{TM}}{U_m^{TM}} \right)^{-1} \approx \left(-1 + \frac{3j}{2} x_0^{-3} \frac{\varepsilon_r + 2}{\varepsilon_r - 1} \right)^{-1}$$

$\hookrightarrow$ real part matters for σ_{ext}

exercise: Under Kerker condition, $\sigma_{ext} \approx x_0^{12} \left(\frac{\mu - 1}{\mu + 2} \right)^4$, optical theorem is fine!

dipolar polarizabilities of small spheres

$$\boxed{\alpha_e = j \frac{6\pi\varepsilon_0}{k_0^3} C_1^{TM} \; ; \quad \alpha_m = j \frac{6\pi\mu_0}{k_0^3} C_1^{TE}}$$

$$\Downarrow$$

$\vec{p} = \alpha_e \vec{E}_{loc}$ with $\alpha_e^{-1} = \frac{k_0^3}{6\pi\varepsilon_0}\left(\frac{V_m^{TM}}{U_m^{TM}} + j\right)$ is the electric dipole polarizability

$\vec{m} = \alpha_m \vec{H}_{loc}$ with $\alpha_m^{-1} = \frac{k_0^3}{6\pi\mu_0}\left(\frac{V_m^{TE}}{U_m^{TE}} + j\right)$ — magnetic — —

7. Resonant scattering

What happens near resonances (plasmonic when $ka \ll 1$, dielectric otherwise)?

Let's use coupled-mode theory:

$$\vec{E}_i = \sum_{m=1}^{+\infty} E_0 j^{-m} \frac{2m+1}{m(m+1)} \left(\vec{M}^{(1)}_{o1m} + j \vec{N}^{(1)}_{e1m} \right) \Rightarrow \text{each spherical harmonic is a port.}$$

Here $^{(1)}$ means $z_m = j_m = \frac{1}{2}\left(h_m^{(1)} + h_m^{(2)}\right)$

$\overset{\rho \to +\infty}{\sim} e^{-jk\rho} \rho^{-1}$ $\quad\quad$ $\overset{\rho \to +\infty}{\sim} \rho^{-1} e^{jk\rho} \rightarrow$ incident power

$\boxed{\text{case 1: One resonance}}$

The power incident on the ℓ'th TE or TM harmonic channel is:

$$P^{(\ell)} = \frac{|E_0|^2}{2\eta_0} \frac{1}{2^2} \frac{2\pi}{k_0^2} (2\ell+1) = \frac{1}{4} \frac{E_0^2}{2\eta_0} \frac{\lambda_0^2}{2\pi} (2\ell+1)$$ (calculation similar to σ_{scat} with $c_m^{TE} = 0$, $c_m^{TM} = 1$)

Assume the incident wave has a frequency ω close to the resonance frequency ω_r of the TE mode $\vec{M}_{omn}$ of the scatterer. It will couple to it and excite it with an amplitude a, normalized such that $|a|^2$ is the energy in the resonance. The S_ℓ^+ port is therefore associated with the power $P^{(\ell)} = |S_\ell^+|^2$.

Let's call γ_r and γ_ℓ the radiative and absorption loss rates.

In the theory of CMT, we have shown that the resonator decay is:

$$\frac{d|a|^2}{dt} = -2\gamma_r |a|^2 - 2\gamma_\ell |a|^2 = P_{scat} + P_{abs} \text{ and so}$$

$$\boxed{\begin{aligned} Q_{rad} &= \frac{\omega_{res}}{2\gamma_r} = \frac{\omega_{res}|a|^2}{P_{scat}} \\ Q_{abs} &= \frac{\omega_{res}}{2\gamma_r} = \frac{\omega_{res}|a|^2}{P_{abs}} \end{aligned}}$$

quality factors of the Mie mode

(CMT valid when $Q_{rad}, Q_{abs} \gg 1$)

The CMT equation for the $m=m-1$: $\frac{da}{dt} = j\omega_r a - (\gamma_\ell + \gamma_r)a + \sqrt{2\gamma_r}\, S_\ell^+$ gives,
for $S_\ell^+ = \sqrt{P^{(\ell)}}\, e^{j\omega t}$

$$|a|^2 = \frac{2\gamma_r |S_\ell^+|^2}{(\omega_r-\omega)^2 + (\gamma_r+\gamma_e)^2} = \frac{2\gamma_r P^{(\ell)}}{(\omega_r-\omega)^2 + (\gamma_r+\gamma_e)^2}$$

$P^{(\ell)} \to \frac{1}{4}\frac{\lambda^2}{2\pi}\frac{|E_0|^2}{2\eta_0}(2\ell+1)$

But since $P_{scat} = 2\gamma_{rad}|a|^2$, $P_{abs} = 2\gamma_{abs}|a|^2$

$$\boxed{\begin{aligned}\sigma_{scat} &= \frac{P_{scat}}{|E_0|^2/2\eta_0} = \frac{\lambda_0^2}{2\pi}(2\ell+1)\frac{\gamma_r^2}{(\omega_r-\omega)^2+(\gamma_r+\gamma_e)^2}\\ \sigma_{abs} &= \frac{\lambda_0^2}{2\pi}(2\ell+1)\frac{\gamma_r\gamma_e}{(\omega_r-\omega)^2+(\gamma_r+\gamma_e)^2}\end{aligned}}$$

single mode resonant scattering cross-sections
- Lorentzians

Rk: 1- Looks like Breit-Wigner formulas for resonant neutron scattering
2- The linewidth of σ_s and σ_{abs} are the same FWHM $= \gamma_r + \gamma_{abs}$
3- Some approximate formulas for dielectric spheres $(\mu=\mu_0, m = m' - jm'', m' \gg m'')$

$\vec{N}_{e1\ell}$ (TM) mode: $\boxed{\omega_{res}^{TM} \simeq \frac{c_0}{ma} z_\ell}$ [1] , z_ℓ is a zero of $u_\ell(\rho) = \rho j_\ell(\rho)$

$\vec{M}_{o1\ell}$ (TE) mode $\boxed{\omega_{res}^{TE} \simeq \frac{c_0}{ma} z_{\ell-1}}$ [1] and $\boxed{Q_{abs} = \frac{m'}{2Km''}}$ [2]

$K \to$ fraction of mode energy inside the sphere

[Refs: [1] R. G. Newton, Scattering theory of waves and particles
[2] PRE 71, 026602 (2005)]

4) $S_\ell^- = \underbrace{C S_\ell^+}_{} + \underbrace{Da}_{\to \text{scattering}}$ $\qquad C_\ell^{TM/TE} = \text{cste} \times \left.\frac{Da}{S_\ell^+}\right|_{TE/TM}$ $\qquad D = j\sqrt{2\gamma_r}$

here even when $a=0$
$\sim h_\ell^{(1)}$ part of the plane wave

$$\Rightarrow C_\ell = \frac{(j\sqrt{2\gamma_r})^2 \cdot \text{cste}}{j(\omega-\omega_r)+\gamma_r+\gamma_L} \quad \text{but} \quad \sigma_{scat} = \frac{\lambda_0^2}{\cancel{2\pi}}\cancel{(2\ell+1)}\, |C_\ell|^2 = \frac{\lambda_0^2}{\cancel{2\pi}}\cancel{(2\ell+1)} \frac{4\gamma_r^2 \text{cst}^2}{j(\omega-\omega_r)+\gamma_r+\gamma_L}$$

$$\Rightarrow \text{cste} = \frac{1}{2} \Rightarrow \boxed{C_\ell^{TE/TM} = \left.\frac{-\gamma_r}{j(\omega-\omega_r)+\gamma_r+\gamma_\ell}\right|_{TE/TM}}$$

scattering coefficients from CMT
$\Rightarrow$ no need to solve the Mie problem when a mode dominates

5) Valid also for non-spherical objects.

$\boxed{\text{Generalization to an arbitrary number of resonant channels}}$

example: overlapping electric and magnetic resonance : 2 modes, 2 ports

$$P_{inc} = \vec{S}_+^\dagger \vec{S}_+ = \frac{1}{4}\frac{E_0^2}{2\eta_0}\underbrace{\sum_m}_{\text{involved TE, TM spherical harmonics}} \frac{\lambda_0^2}{2\pi}(2\ell_m+1) \qquad \boxed{\vec{S}_+ = \sqrt{\frac{1}{4}\frac{E_0^2}{2\eta_0}\frac{\lambda_0^2}{2\pi}}\,\underbrace{[\sqrt{2\ell_1+1};\dots;\sqrt{2\ell_m+1}]^T}_{\equiv\ \vec{L}}}$$

$$\text{Reminder : } \vec{S}_- = \underbrace{C\vec{S}_+}_{\text{direct path = not scattering}} + \overbrace{D\vec{a}}^{\text{scattering}} \Rightarrow P_{scat} = (D\vec{a})^\dagger D a = \vec{a}^\dagger D^\dagger D a = 2\vec{a}^\dagger \Gamma_R \vec{a}$$

but $\vec{a} = \left(j(\omega\mathbb{1}_m - \Omega) + \Gamma\right)^{-1} K^T \vec{S}_+$ (eq (4) in general CMT)

$$\Rightarrow P_{scat} = \vec{S}_+^\dagger \underbrace{K}_{=D\ (\text{TRS})}{}^{T\dagger}\left(-j(\omega\mathbb{1}_m - \Omega) + \Gamma\right)^{-1} D^\dagger D \left(j(\omega\mathbb{1}_m - \Omega) + \Gamma\right)^{-1} K^T \vec{S}_+$$

$$\Rightarrow \boxed{\sigma_{scat} = \frac{P_{scat}}{\frac{1}{2}\frac{E_0^2}{\eta_0}} = \frac{1}{4}\frac{\lambda_0^2}{2\pi}\vec{L}^\dagger D^* \left[-j(\omega\mathbb{1}_m - \Omega) + \Gamma\right]^{-1} D^\dagger D \left[j(\omega\mathbb{1}_m + \Omega) + \Gamma\right]^{-1} D^T \vec{L}}$$

$$\sigma_{abs} = \frac{\vec{S}_+^\dagger \vec{S}_+ - \vec{S}_-^\dagger \vec{S}_-}{\frac{1}{2}\frac{E_0^2}{\eta_0}} = \frac{\vec{S}_+^\dagger(\mathbb{1}_m - S^\dagger S)\vec{S}_+}{\frac{1}{2}\frac{E_0^2}{\eta_0}} \Rightarrow \boxed{\sigma_{abs} = \frac{1}{4}\frac{\lambda_0^2}{2\pi}\vec{L}^\dagger(\mathbb{1} - S^\dagger S)\vec{L}}$$

[Exercise: for 1 port 1 mode, we retrieve the previous formulae]

Scattering coefficients:

$$\boxed{\underbrace{\hat{C}}_{\text{matrix of scattering coefficients}} = \frac{1}{2}D\left(j(\omega\mathbb{1}_m - \Omega) + \Gamma\right)^{-1}K^T = \frac{1}{2}(\underbrace{S}_{\text{CMT scattering matrix}} - \underbrace{C}_{\text{CMT direct path matrix}})}$$

generalized Mie coefficients

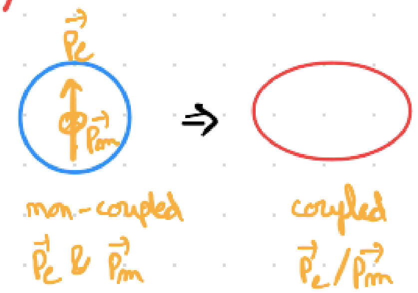
No direct path C in σ_{scat} ⇒ Fano requires two modes

More general than just spherical object: ex for C_1^{TE} and C_1^{TM}, 2 modes

polarisation conversion (oblong)

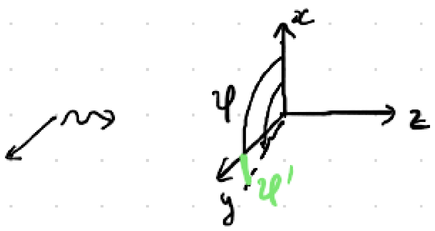
$$\hat{C} = \begin{pmatrix} \boxed{C_1^{TE}} & \boxed{C_1^{TM \to TE}} \\ \boxed{C_1^{TE \to TM}} & \boxed{C_1^{TM}} \end{pmatrix}$$

Mie (spherical)



9. Other incident polarization

$\vec{E}_{inc} = E_0 \vec{y} e^{-jk_0 z}$? For a sphere, no difference except that what should now replace φ in all formulas is $\varphi' = \varphi - \pi/2$.



$$\boxed{\vec{X}(\theta, \varphi) = \vec{Y}(\theta, \varphi + \frac{\pi}{2})}$$